Thm 2.2 $\quad$ F.S. $\hookrightarrow$ BFS.

pf: $\quad A\vec{x} = \vec{b}, \quad \vec{x} \geq 0 \quad$ "rank(A) = m"

$$m\left\{\left(\underbrace{\vec{a}_1, \vec{a}_2 \ldots \quad \vec{a}_n}_{n}\right)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \vec{b} \qquad x_i \geq 0 \ \forall i.\right.$$

$$(\vec{a}_1, \vec{a}_2, \ldots \vec{a}_p, \vec{a}_{p+1} \ldots \vec{a}_n)\begin{pmatrix} x_1 \\ \vdots \\ x_p \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \vec{b}$$

Can be "p = n"

$$\Leftrightarrow \sum_{i=1}^{p} x_i \vec{a}_i = \vec{b} \qquad (1)$$

Case 1: $\{\vec{a}_1, \ldots, \vec{a}_p\}$ linear independent $\quad \vec{a}_i \in \mathbb{R}^m$

Case 2: $\{\vec{a}_1, \ldots \vec{a}_p\}$ linear dependent

Case 1: $\{\vec{a}_1, \ldots \vec{a}_p\}$ are linear independent.

(i) $p = m$; (ii) $p < m$; (iii) $p > m$ $\rightarrow$ this is impossible

(i): $$\Big(\overbrace{\underbrace{\vec{a}_1 \ldots \vec{a}_m}_{\text{li}}}^{\text{invertible}} \ldots\Big)\begin{pmatrix} x_1 \\ \vdots \\ x_m \\ 0 \\ \vdots \end{pmatrix} = \vec{b}$$

B.F.S

(ii): $$\Big(\overbrace{\underbrace{\vec{a}_1 \ldots \vec{a}_p\, \vec{a}_{p+1} \ldots \vec{a}_m}_{\text{li}}}^{\text{invertible}}\Big)\begin{pmatrix} x_1 \\ \vdots \\ x_p \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{m}^{p} = \vec{b}$$

BFS (degenerate)

Case 2 $\{\vec{a}_1, \dots \vec{a}_p\}$ ld.

FS $\rightarrow$ eliminate $\vec{a}_i$ one by one $\rightarrow$ remaining FS $\{\vec{a}_i\}$ l.i.

$\because \{\vec{a}_1, \dots \vec{a}_p\}$ ld

$\exists\ \alpha_i \in \mathbb{R}$ not all zero st

$$\alpha_1 \vec{a}_1 + \alpha_2 \vec{a}_2 + \dots + \alpha_p \vec{a}_p = \vec{0}$$

$$A \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_p \\ 0 \\ \vdots \end{pmatrix} = \vec{0}$$

non zero

Suppose $\alpha_r \neq 0$

$$\vec{a}_r = -\frac{\alpha_1 \vec{a}_1}{\alpha_r} - \frac{\alpha_2 \vec{a}_2}{\alpha_r} - \dots - \frac{\alpha_{r-1}}{\alpha_r}\vec{a}_{r-1} - \frac{\alpha_{r+1}}{\alpha_r}\vec{a}_{r+1} - \dots - \frac{\alpha_p}{\alpha_r}\vec{a}_p \quad (2)$$

(2) $\rightarrow$ (1)

$$A \begin{pmatrix} x_1 - \frac{\alpha_1}{\alpha_r} x_r \\ \vdots \\ x_i - \frac{\alpha_i}{\alpha_r} x_r \\ \vdots \\ 0 \\ x_p - \frac{\alpha_p}{\alpha_r} x_r \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{array}{l} \geq 0 \\ \\ \geq 0 \\ \\ \leftarrow r^{th} \\ \geq 0 \quad p \\ \\ \\ m \end{array} = \vec{b}$$

One more variable set to zero

Is this feasible $\Leftrightarrow$

$$x_i - \frac{\alpha_i}{\alpha_r} x_r \geq 0 \quad \forall i \ ??$$

Choice 1 Choose $\alpha_r > 0$

s.t. $$\frac{x_r}{\alpha_r} = \min_{1 \le j \le p} \left\{ \frac{x_j}{\alpha_j} \;\middle|\; \alpha_j > 0 \right\} \qquad (3)$$

$$\iff \frac{x_r}{\alpha_r} \le \frac{x_j}{\alpha_j} \quad \forall j \text{ s.t. } \alpha_j > 0$$

$$\iff \underbrace{x_j - \alpha_j \frac{x_r}{\alpha_r}}\ \ge 0 \quad \forall j \text{ s.t. } \alpha_j > 0$$

What if $\alpha_j < 0$

$$\underset{\overset{\vee\!\shortmid}{0}}{x_j} - \overset{\overset{0}{\vee\!\shortmid}}{\alpha_j}\, \frac{\overset{\overset{0}{\wedge\!\shortmid}}{x_r}}{\underset{\overset{\vee}{0}}{\alpha_r}} \ge 0$$

Choice 2 Choose $\alpha_r < 0$

st $$\frac{x_r}{\alpha_r} = \max_{1 \le j \le p} \left\{ \frac{x_j}{\alpha_j} \;\middle|\; \alpha_j < 0 \right\}$$

\#

$$\begin{pmatrix} x_1 \\ \vdots \\ x_p \\ 0 \\ \vdots \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ 0 \\ \tilde{x}_{r+1} \\ \vdots \\ \tilde{x}_p \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{ F.S} \qquad \xrightarrow{\text{repeat}}$$

$0 \leftarrow$ rth coordinate

p nonzero FS $\qquad$ p-1 nonzero FS

$p \le m$ and $\{\vec{a}_1 \ldots \vec{a}_p\}$ l indepe.

Case 1

Eg 4.4

x_1, x_2, x_3
$\quad\ \ \|$
$2 \quad 3 \quad 1$

$\alpha_1 \quad \alpha_2 \quad \alpha_3$
$\quad\ \ \|$
$1 \quad 2 \quad -1$

$\boxed{\text{Choice 1}}$ $\alpha_r > 0$

$\frac{x_1}{\alpha_1} \quad \frac{x_2}{\alpha_2} \quad \frac{x_3}{\alpha_3}$ $\quad \alpha_i > 0$

$\|\quad\ \|$

$2 > \frac{3}{2}$

Should choose x_2 to eliminate $\rightarrow$ FS

Choice 2

$\frac{x_1}{\alpha_1} \quad \frac{x_2}{\alpha_2} \quad \frac{x_3}{\alpha_3}$ $\quad \alpha_3 < 0$

$\|$

$\frac{1}{-1}$.

Should choose x_3 to eliminate $\rightarrow$ FS.

Eliminate x_2:

$$2\vec{a}_2 = \vec{a}_3 - \vec{a}_1 \Rightarrow \vec{a}_2 = \frac{1}{2}\vec{a}_3 - \frac{1}{2}\vec{a}_1$$

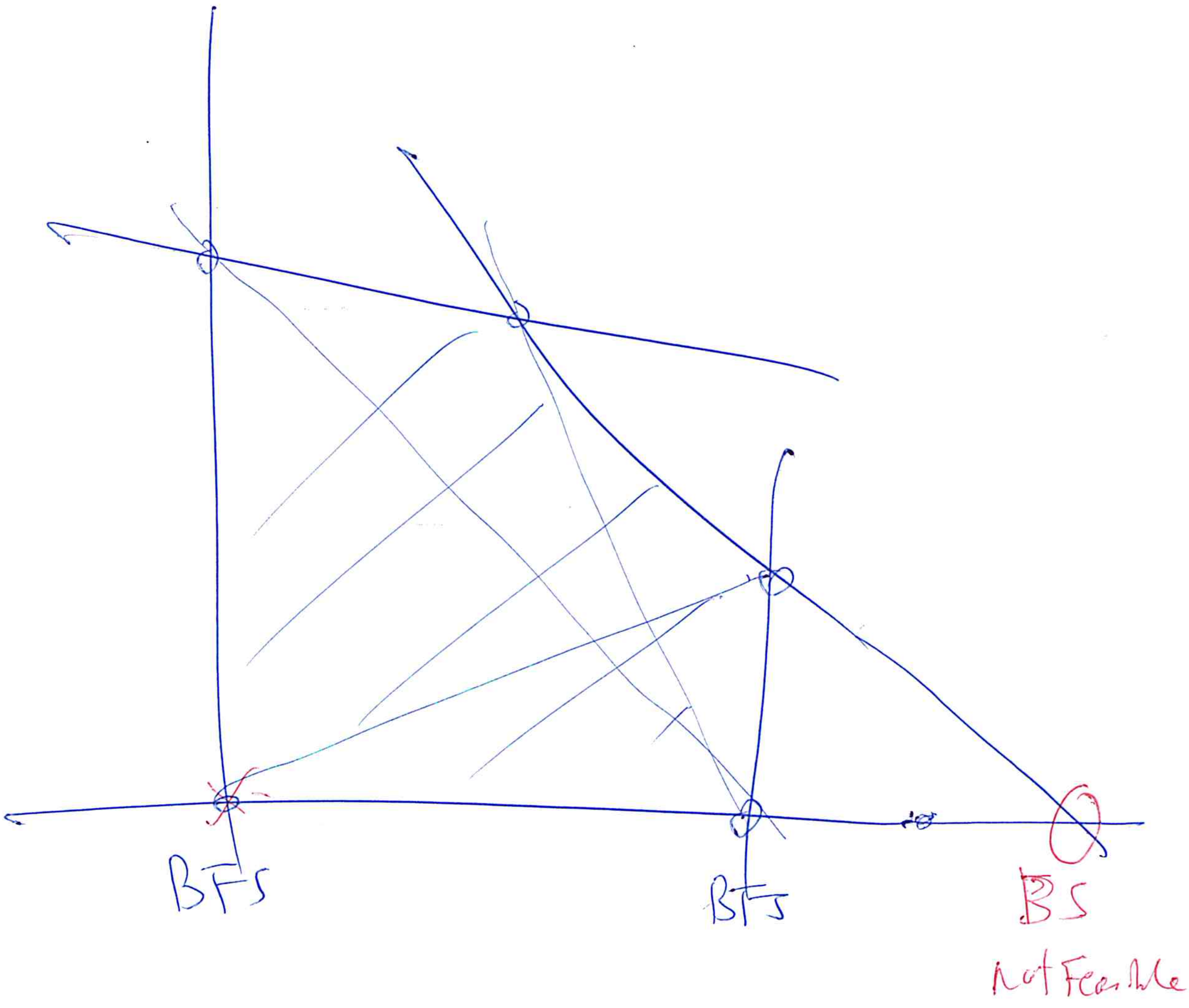
$$\therefore \quad 2\vec{a}_1 + 3\vec{a}_2 + 1\vec{a}_3 = \vec{b}$$

$$\Rightarrow 2\vec{a}_1 + 3\left(\frac{1}{2}\vec{a}_3 - \frac{1}{2}\vec{a}_1\right) + \vec{a}_3 = \vec{b}$$

$$\Rightarrow \frac{1}{2}\vec{a}_1 + \frac{5}{2}\vec{a}_3 = \vec{b}$$

$$\Rightarrow A \cdot \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{5}{2} \end{pmatrix} = \vec{b}$$

$\begin{pmatrix} 1/2 \\ 0 \\ 5/2 \end{pmatrix}$ is a basic feasible solution.


BFS
BFS
BS
Not Feasible